NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 1
Assessment Task 2
Term 1, 2015

Name: ____________________________________ **Mathematics Class:__________**

Student Number: ________________

Time Allowed: **50 mins + 2 mins reading time**
Total Marks:
Section I - **Pages 2 - 4**
5 Marks
Attempt Questions 1- 5
Answer on Multiple Choice Answer Sheet

Section II - **Pages 5 - 7**
30 Marks
Attempt Questions 6 - 8
Show all necessary working

Question	Q1	Q2-4	Q5	Q6	Q7	Q8	Total
PE3					/3		/3
HE2			/1	/3		/4	/8
HE6	/1			/6	/3	/3	/13
HE7		/3			/4	/4	/11
							/35

Section I

5 Marks
Attempt Questions 1 – 5
Allow about 7 minutes.
Use the multiple choice answer sheet for questions 1 - 5

1. Using $u = x^2 + 1$, which of the following is equivalent to $\int_0^2 \frac{xdx}{\sqrt{x^2+1}}$?

(A) $\frac{1}{2}\int_1^5 u^{-\frac{1}{2}} du$

(B) $2\int_1^5 u^{-\frac{1}{2}} du$

(C) $\frac{1}{2}\int_0^2 u^{-\frac{1}{2}} du$

(D) $2\int_0^2 u^{-\frac{1}{2}} du$

2. What is the value of $\sum_{n=0}^{\infty} 5 \times \left(\frac{1}{4}\right)^{n-1}$?

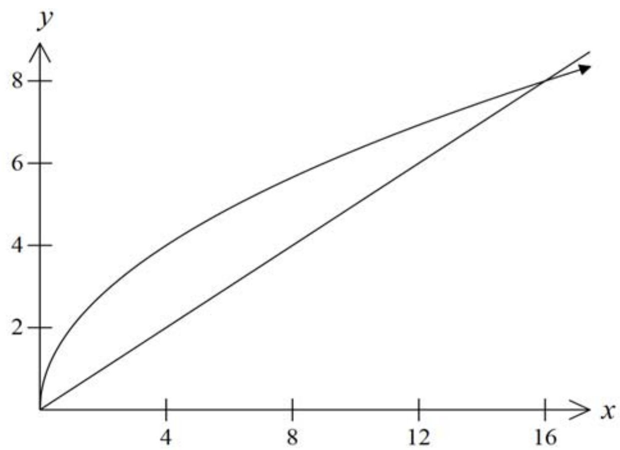
(A) 0

(B) $\frac{20}{3}$

(C) $\frac{80}{3}$

(D) $\frac{15}{4}$

3. The diagram below shows the graph of $y = 2\sqrt{x}$ and $y = \frac{x}{2}$.



Which of the following is the correct expression for the volume of the solid of revolution when the area between the curve $y = 2\sqrt{x}$ and $y = \frac{x}{2}$ is rotated around the x-axis?

(A) $V = \int_0^{16} \left(4x - \frac{x^2}{4}\right) dx$

(B) $V = \int_0^{16} \left(2\sqrt{x} - \frac{x}{2}\right)^2 dx$

(C) $V = \pi \int_0^{16} \left(2\sqrt{x} - \frac{x}{2}\right)^2 dx$

(D) $V = \pi \int_0^{16} \left(4x - \frac{x^2}{4}\right) dx$

4. Suppose that f is a continuous function and that $\int_1^5 f(x)dx = -6$ and $\int_2^5 3f(x)dx = 6$.

What is the value of $\int_1^2 f(x)dx$?

(A) -8

(B) -12

(C) 8

(D) 12

5. Given that $n > 1$, which of the following sum representations is incorrect?

(A) $(1-x)+(1-x^2)+(1-x^3)+...+(1-x^n) = \sum_{r=0}^{n-1} \left[(1-x^{n-r})\right]$

(B) $\frac{1}{2!}+\frac{2}{3!}+...+\frac{n-1}{n!} = \sum_{r=2}^{n} \frac{r-1}{r!}$

(C) $1^3+2^3+...+n^3 = \sum_{r=1}^{n-1}\left[(n-r)^3\right]$

(D) $x+(x-1)+(x-2)+...+(x-n) = \sum_{r=1}^{n+1}(x-r+1)$

End of Section I

Section II

30 Marks
Attempt Questions 6 – 8
Allow about 43 minutes.

Answer each question in a SEPARATE writing booklet. Extra writing paper is available.

In Questions 6 – 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (9 Marks) Use a separate writing booklet.

(a) Find $\int 2x\sqrt{x^2+9}\,dx$ using $u = x^2+9$. **3**

(b) Use the substitution $t = u^2 - 1$ to evaluate $\int_0^1 \frac{t}{\sqrt{1+t}}\,dt$ **3**

(c) Prove by mathematical induction that $4^n + 15n - 1$ is divisible by 9 for all positive integers n. **3**

End of Question 6

Question 7 (10 marks) Use a separate writing booklet

(a) (i) Show that $\dfrac{d}{dx}\left(\dfrac{2x^2+1}{3x^2-4}\right)=-\dfrac{22x}{\left(3x^2-4\right)^2}$ **1**

(ii) Hence, evaluate $\displaystyle\int_0^1 \frac{x}{\left(3x^2-4\right)^2}\,dx$. **2**

(b) What is the least integer value of n for which **3**

$1+3+3^2+...+3^{n-1}>10^4$?

(c) (i) Find the nth term of each of the series below: **2**

$A_n=3+6+12+...$

$B_n=2+7+12+...$

(ii) Deduce, by considering the terms of the series A_n and B_n, the tenth term of the following series $C_n=10+26+48+...$ **2**

End of Question 7

Question 8 (11 marks) Use a separate writing booklet

(a) Let $f(x)$ be an odd function, where p is a positive constant.

(i) Prove that $\int_0^{2p} f(x-p)dx = 0$, using the substitution $u = x - p$. **2**

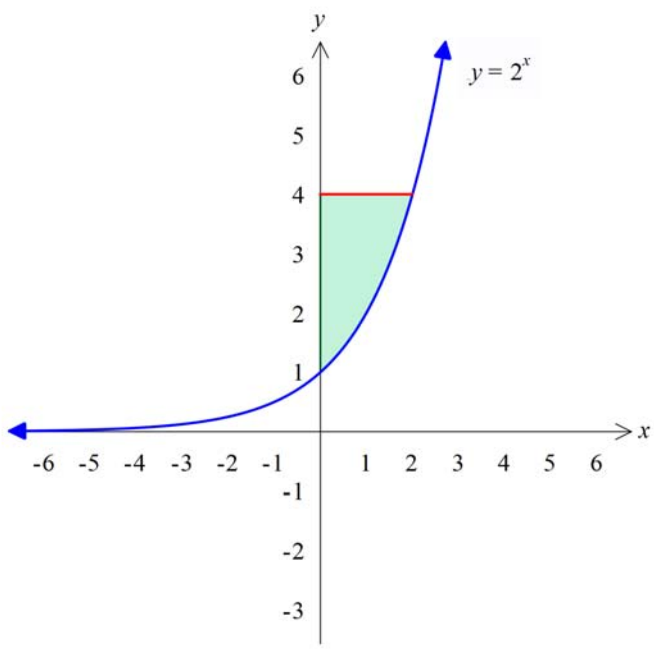
(ii) Hence evaluate $\int_0^{2p} \left[f\left(x-p\right)+q \right] dx$, where q is a constant. **1**

(b) (i) Show that for $k \geq 0$, $\left(k+1\right)^5 - (k+1)^4 \geq k^5$. **1**

(ii) Hence, use mathematical induction to prove for all integers $n \geq 1$, **3**

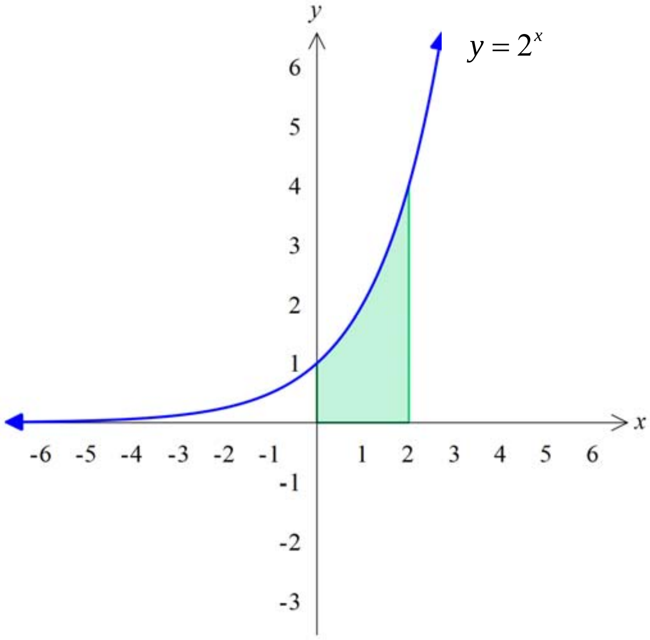
$$1^4 + 2^4 + ... + n^4 \leq n^5.$$

(c) (i) The region bounded by the curve $y = 2^x$, the y-axis and the line $y = 4$ is rotated around the y-axis. Use Simpson's Rule with three function values to find the volume of the solid generated. **3**
Give your answer correct to 3 decimal places.



Question 8 continues over the page

(ii) Hence, find the volume generated when the shaded area in the graph below is rotated around the y-axis. **1**



End of Examination

NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 1
Assessment Task 2
Term 1, 2015

SOLUTIONS

Section I

1. Using $u = x^2 + 1$, which of the following is equivalent to $\int_0^2 \frac{xdx}{\sqrt{x^2+1}}$?

Answer: A

$$\int_0^2 \frac{xdx}{\sqrt{x^2+1}} = \int_1^5 \frac{1}{\sqrt{u}}\frac{du}{2}$$
$$= \frac{1}{2}\int_1^5 u^{-\frac{1}{2}}du$$

$u = x^2 + 1$
$du = 2xdx$
when $x = 0,\ u = 1$
when $x = 2,\ u = 5$

2. What is the value of $\sum_{n=0}^{\infty} 5\times\left(\frac{1}{4}\right)^{n-1}$?

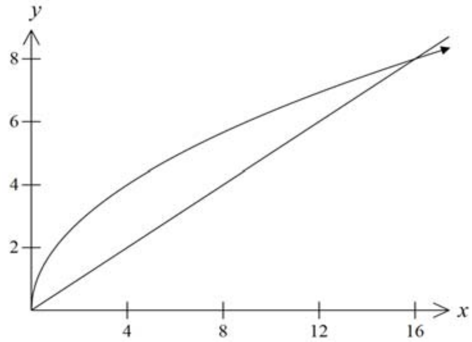
Answer: C

$$20 + 5 + \frac{5}{4} + \ldots$$

geometric series with $a = 20,\ r = \frac{1}{4}$

$$S_\infty = \frac{20}{1-\frac{1}{4}}$$
$$= \frac{80}{3}$$

3. The diagram below shows the graph of $y = 2\sqrt{x}$ and $y = \frac{x}{2}$.



Which of the following is the correct expression for the volume of the solid of revolution when the area between the curve $y = 2\sqrt{x}$ and $y = \frac{x}{2}$ is rotated around the *x*-axis?

Answer: D

$$V = \pi\int \left(2\sqrt{x}\right)^2 - \left(\frac{x}{2}\right)^2 dx$$

$$= \pi\int \left(4x - \frac{x^2}{4}\right)dx$$

4. Suppose that f is a continuous function and that $\int_1^5 f(x)dx = -6$ and $\int_2^5 3f(x)dx = 6$.

What is the value of $\int_1^2 f(x)dx$?

Answer: A

$$\int_1^5 f(x)dx = \int_1^2 f(x)dx + \int_2^5 f(x)dx$$

$$\therefore \int_1^2 f(x)dx = -6 - 2$$

$$= -8$$

5. Given that $n > 1$, which of the following sum representations is incorrect?

Answer: C

$$\sum_{r=1}^{n-1}\left[(n-r)^3\right] = (n-1)^3 + (n-2)^3 + \ldots + (1)^3$$

$$\neq n^3 + (n-1)^3 + (n-2)^3 + \ldots + (1)^3$$

Section II
Question 6

(a) Find $\int 2x\sqrt{x^2+9}\,dx$ using $u = x^2+9$. **3**

$$\int 2x\sqrt{x^2+9}\,dx = \int u^{\frac{1}{2}}\,du$$
$$= \left[2\frac{u^{\frac{3}{2}}}{3}\right] + C$$
$$= \frac{2}{3}\left(x^2+9\right)^{\frac{3}{2}} + C$$

$u = x^2+9$
$du = 2x\,dx$

Comments:
This question was generally well done by most students. But some forgot to sub back to x and some forgot to + C at the end.

(b) Use the substitution $t = u^2-1$ to evaluate $\int_0^1 \frac{t}{\sqrt{1+t}}\,dt$ **3**

$$\int_0^1 \frac{t}{\sqrt{1+t}}\,dt = 2\int_1^{\sqrt{2}} \frac{u^2-1}{u}\times u\,du$$
$$= 2\int_1^{\sqrt{2}} (u^2-1)\,du$$
$$= 2\left[\frac{u^3}{3} - u\right]_1^{\sqrt{2}}$$
$$= 2\left[\left(\frac{2\sqrt{2}}{3} - \sqrt{2}\right) - \left(\frac{1}{3} - 1\right)\right]$$
$$= 2\left[-\frac{\sqrt{2}}{3} + \frac{2}{3}\right]$$
$$= \frac{2\left(2-\sqrt{2}\right)}{3}$$

$t = u^2-1$
$dt = 2u\,du$
when $t = 0,\ u = 1$
when $t = 1,\ u = \sqrt{2}$

Comments:
The question was not done well. For substitution into an integral, all students remembered to convert t values to u but many students evaluated u incorrectly. (e.g. $u = -1$ for $t = 0$; $u = 0$ for $t = 1$ etc.).
Some evaluated final answer without showing substitution.

(c) Prove by mathematical induction that $4^n + 15n - 1$ is divisible by 9 for all positive integers n. **3**

Prove for $n = 1$.
$4^1 + 15(1) - 1 = 18$ which is divisible by 9.
$\therefore$ true for $n = 1$.

Assume true for $n = k$.

$4^k + 15k - 1 = 9M$ where M is an integer.

Thus, $4^k = 9M - 15k + 1$

Prove true for $n = k + 1$.

i.e. need to prove $4^{k+1} + 15(k+1) - 1 = 9N$ where N is an integer.

$$\begin{aligned}\text{L.H.S.} &= 4^{k+1} + 15(k+1) - 1 \\ &= 4.4^k + 15k + 14 \\ &= 4(9M - 15k + 1) + 15k + 14 \text{ using the assumption} \\ &= 36M - 45k + 18 \\ &= 9(4M - 5k + 2) \\ &= 9N \text{ since } (4M - 5k + 2) \text{ is an integer.}\end{aligned}$$

$\therefore$ by the principle of mathematical induction, $4^n + 15n - 1$ is divisible by 9 for all $n \geq 1$.

Comments:
The majority of students knew the steps very well and took care with the setting out. Some forgot to write " " is an integer (– ½ marks).The final conclusion need only be short: " Therefore by Mathematical Induction the result is true for ______ "

Question 7

(a) (i) Show that $\dfrac{d}{dx}\left(\dfrac{2x^2+1}{3x^2-4}\right) = -\dfrac{22x}{(3x^2-4)^2}$ 1

$$\begin{aligned}\frac{d}{dx}\left(\frac{2x^2+1}{3x^2-4}\right) &= \frac{(3x^2-4).4x - (2x^2+1).6x}{(3x^2-4)^2} \\ &= \frac{12x^2 - 16x - 12x^2 - 6x}{(3x^2-4)^2} \\ &= -\frac{22x}{(3x^2-4)^2}\end{aligned}$$

(ii) Hence, evaluate $\displaystyle\int_0^1 \frac{x}{(3x^2-4)^2}\,dx$. **2**

$$\begin{aligned}\int_0^1 \frac{x}{(3x^2-4)^2}\,dx &= -\frac{1}{22}\int_0^1 \frac{-22x}{(3x^2-4)^2}\,dx \\ &= -\frac{1}{22}\left[\frac{2x^2+1}{3x^2-4}\right]_0^1 \\ &= -\frac{1}{22}\left[\left(\frac{3}{-1}\right) - \left(\frac{1}{-4}\right)\right] \\ &= -\frac{1}{22} \times -\frac{11}{4} \\ &= \frac{1}{8}\end{aligned}$$

Comments:

i) A few students could not recall the quotient rule correctly. Using the product rule, added time and effort. Anyone who dropped the denominator in a line of working was penalised. Most students earned the full mark.

ii) A small minority of students did not substitute the limits, leaving the answer as a function of x. A small number of students could not handle the 22 or the minus sign correctly and some made the mistake of assuming that every time you substitute zero, you get zero. Most students who didn't make these errors did take the $-\frac{1}{22}$ outside the integral (and the square brackets) making the ensuing calculation simpler.

(b) What is the least integer value of n for which **3**

$$1+3+3^2+...+3^{n-1}>10^4 \text{ ?}$$

Geometric series $a=1,\ r=3.\quad S_n=\frac{1(3^n-1)}{3-1}$

Need to solve: $\frac{3^n-1}{2}>10\ 000$

$$3^n>20\ 001$$

Taking logs of b.s $\log_{10}3^n>\log_{10}20001$

$$n\log_{10}3>\log_{10}20001$$

$$n>\frac{\log_{10}20001}{\log_{10}3}$$

$$n>9.01458...$$

$$\therefore n=10$$

Comments:

A small minority only considered the last term, making no attempt to take a sum. The maximum mark they could achieve was 1/3. Some thought that there were $n-1$ terms in the expression instead of n. Students are encouraged to use $S_n=\frac{a\left(r^n-1\right)}{r-1}$ when $r>1$, and $S_n=\frac{a\left(1-r^n\right)}{1-r}$ when $r<1$ in order to avoid errors. Students needed to interpret from $n>9.01$ that $n=10$ is the least value. Students who guessed the final answer by trial and error without using logarithms did not receive full marks.

(c) (i) Find the nth term of each of the series below: **2**

$A_n=3+6+12+...$

$B_n=2+7+12+...$

$A_n:T_n=3\times2^{n-1}$

$B_n:T_n=2+(n-1).5$

(ii) Deduce, by considering the terms of the series A_n and B_n, the tenth term of the following series $C_n=10+26+48+...$ **2**

$C_n=2\left(A_n+B_n\right)$

$\therefore$ for the series C_n,

$T_{10} = 2\times\left(3\times 2^9 + 2 + 9\times 5\right)$

$=3166$

Comments:
(i) The nth term must by definition involve n.
(ii) Students needed to see the link to the earlier series.
Otherwise, part (c) was well done.

Question 8

(a) Let $f(x)$ be an odd function, where p is a positive constant.

(i) Prove that $\displaystyle\int_0^{2p} f(x-p)dx = 0$, using the substitution $u = x - p$. **2**

$$\int_0^{2p} f(x-p)dx = \int_{-p}^{p} f(u)du$$

$= 0$ since $f(x)$ is an odd function.

$u = x - p.$
$du = dx$
when $x = 0,\ u = -p$
when $x = 2p,\ u = p$

(ii) Hence evaluate $\displaystyle\int_0^{2p}\left[f\left(x-p\right)+q\right]dx,$ where q is a constant. **1**

$$\int_0^{2p}\left[f\left(x-p\right)+q\right]dx, \ = \int_0^{2p}\left[f\left(x-p\right)\right]dx + \int_0^{2p} q dx$$

$= 0 + [qx]_0^{2p}$

$= 2pq$

Comments: Mostly well done.

(i) Some students tried to integrate $f(u)$ and incorrectly got something like $\dfrac{u^2}{2}$ and lost a mark. Some students wrote the primitive as $F(u)$. These students then needed to make the argument that $F(x)$ is even since $f(x)$ is odd and then use the defn. of an even function.

(b) (i) Show that for $k \ge 0$, $\left(k+1\right)^5 - (k+1)^4 \ge k^5$. **1**

L.H.S $= (k+1)^5 - (k+1)^4$

$= (k+1)^4(k+1-1)$

$= (k+1)^4.k$

$\ge k^4.k$ since $k+1 \ge k$ and $k \ge 0$.

$= k^5$

$\therefore (k+1)^5 - (k+1)^4 \ge k^5$

Comments:

Students lost $\dfrac{1}{2}$ a mark for a poor explanation in a question that is a "Show that..".

This is not a mathematical induction question.
Statement needs to be shown true for all $k \ge 0$, not just particular values of k.

(ii) Hence, use mathematical induction to prove for all integers $n \geq 1$, **3**

$$1^4 + 2^4 + ... + n^4 \leq n^5.$$

Prove for $n = 1$.

L.H.S $= 1^4$

$= 1$

R.H.S $= 1^5$

$= 1$

L.H.S. $\leq$ R.H.S

$\therefore$ true for $n = 1$.

Assume true for $n = k$.

$1^4 + 2^4 + ... + k^4 \leq k^5$

Prove true for $n = k + 1$.

i.e. need to prove $1^4 + 2^4 + ... + k^4 + (k+1)^4 \leq (k+1)^5$

L.H.S. $= 1^4 + 2^4 + ... + k^4 + (k+1)^4$

$\leq k^5 + (k+1)^4$ using the assumption.

$\leq (k+1)^5$ from (i) $k^5 \leq (k+1)^5 - (k+1)^4$

$\therefore k^5 + (k+1)^4 \leq (k+1)^5$

$=$ R.H.S.

$\therefore$ by the principle of mathematical induction, $1^4 + 2^4 + ... + n^4 \leq n^5$ for all integers $n \geq 1$.

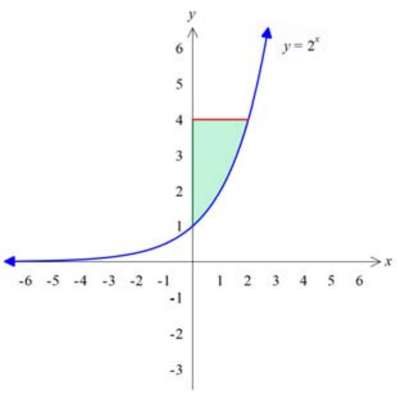
Comments:

$-\frac{1}{2}$ mark where there was poor explanation in the proof.

For inequalities, students are advised to start at LHS and work through proof to show $\leq$RHS. Those that didn't usually did not achieve full marks. i.e writing $1^4 + 2^4 + ... + k^4 + (k+1)^4 \leq (k+1)^5$, $\therefore k^5 + (k+1)^4 \leq (k+1)^5$ is incorrect.

(c) (i) The region bounded by the curve $y = 2^x$, the y-axis and the line $y = 4$ is rotated around the y-axis. Use Simpson's Rule with three function values to find the volume of the solid generated. **3**

Give your answer correct to 3 decimal places.



$h = \frac{4-1}{2}$

$= \frac{3}{2}$

$x = \log_2 y$

$$V = \pi \int_1^4 (\log_2 y)^2 \, dy$$

$$\approx \pi \times \frac{1.5}{3}\left(0 + 4(\log_2 2.5)^2 + 4\right)$$

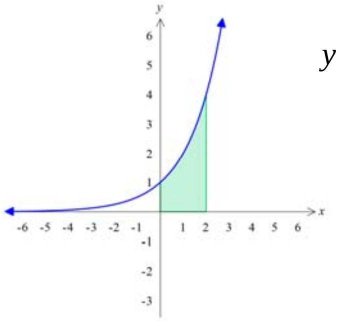
$$= \pi \times \frac{1.5}{3}\left[4\left(\frac{\log_{10} \frac{5}{2}}{\log_{10} 2}\right)^2 + 4\right]$$

$$\approx 17.263 \text{ units}^3$$

y	$x = \log_2 y$	x^2	w	$w.x^2$
1	0	0	1	0
2.5	$\log_2 2.5$	$(\log_2 2.5)^2$	4	$(\log_2 2.5)^2$
4	2	4	1	4

$\sum w.x^2 = 10.98997...$

(ii) Hence, find the volume generated when the shaded area in the graph below is rotated around the y-axis. **1**



$y = 2^x$

$$V = \pi \times 2^2 \times 4 - 17.263$$

$$\approx 33.002 \text{ units}^3$$

Comments:
(i) A minority of students did not make x the subject first while others made errors when expressing in logarithmic form.
Setting out was mostly not good. Very few students explained how they evaluated $(\log_2 2.5)^2$ using change of base.
Don't round your answer until the end.